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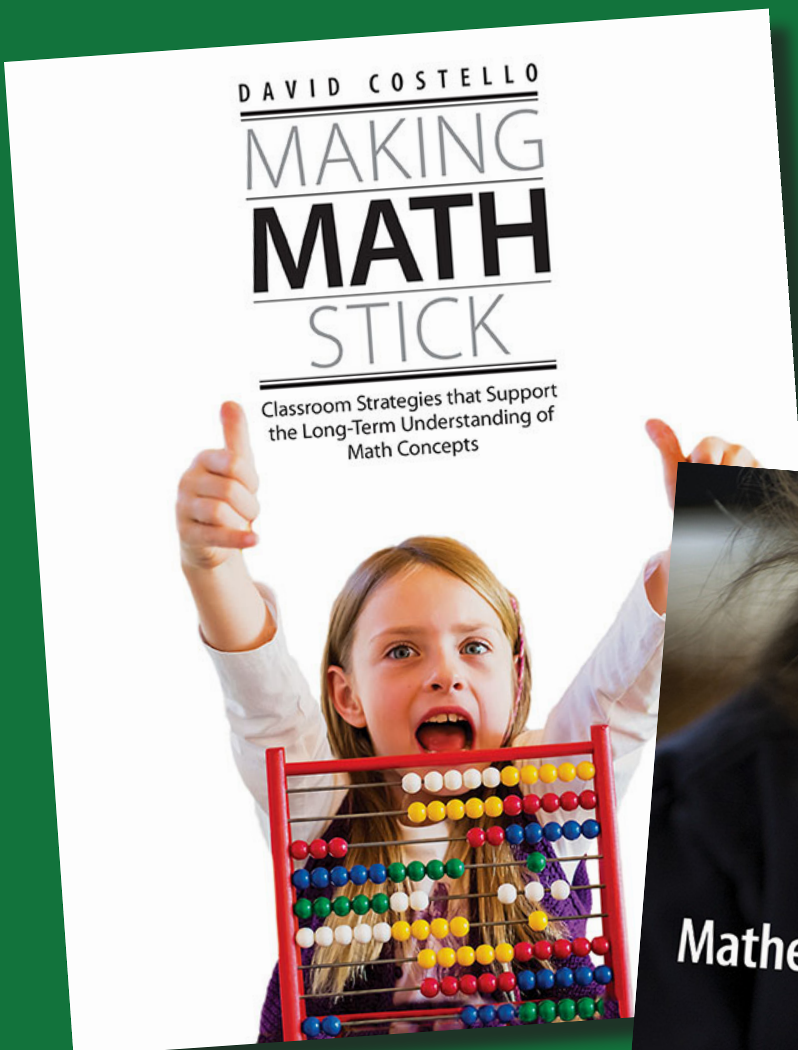
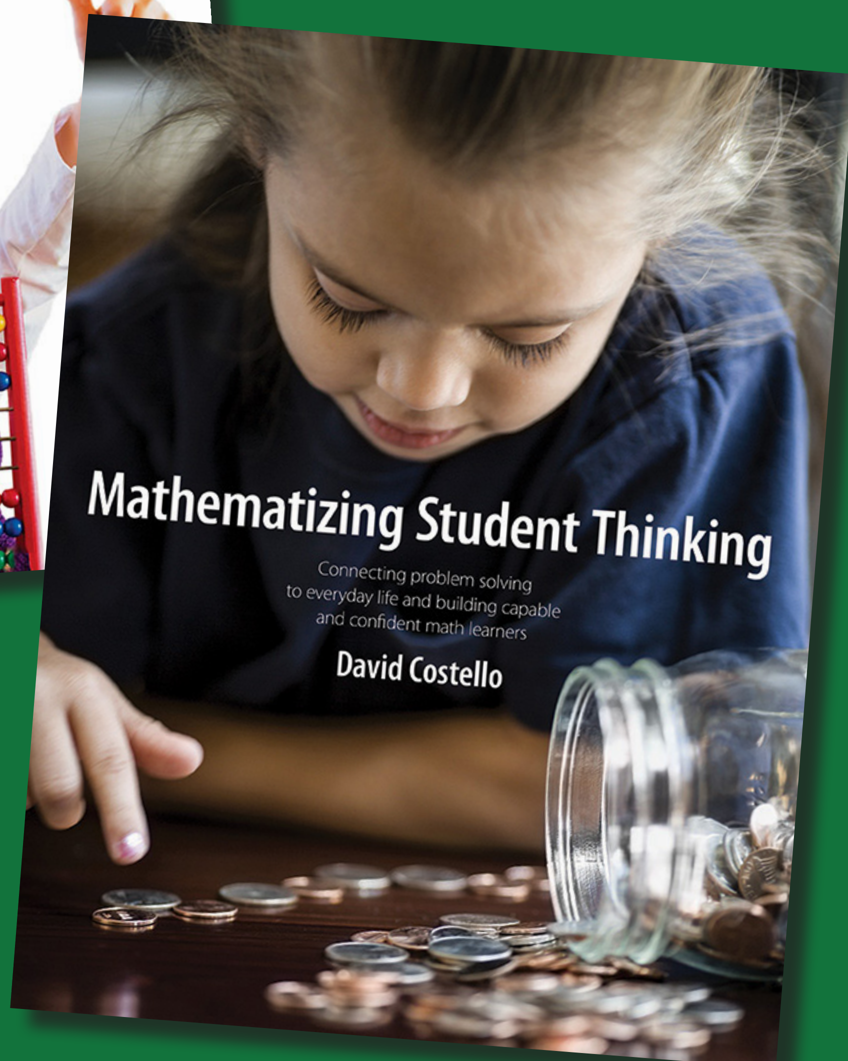
Math & Numeracy





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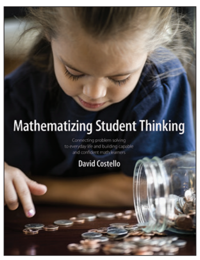
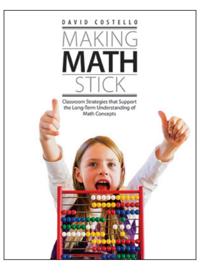
We are pleased to offer sample chapters from two of our newest and bestselling titles on math and numeracy—a peek inside two key resources. This sampler is provided as a PDF, which can be easily read on PC or Apple computers, as well as any number of commercially available e-readers and mobile devices. You can also copy the file to multiple devices, or print the reproducible pages for classroom use.

As a PDF, this sampler is also easily searchable—if you are looking for information on a particular topic, just enter a word or phrase into the Adobe Reader search box, and you'll find every place that word or phrase appears in the document. If any (or all!) of these chapters intrigue you enough to want to read further, simply click on the cover page within the sampler to be taken to the appropriate "buy" page at **www.pembrokepublishers.com**. We offer each title in print and eBook versions—or package the two versions for only $10 more than the print price. And don't forget, any purchase made from Pembroke is eligible for our **10% education discount**!

We hope that you enjoy this special look into Pembroke's practical books for teaching and learning.



How can you broaden student thinking and help them develop their independence and confidence as problem solvers? Real-life problems are a remarkable tool to stretch student thinking and help them develop a deeper understanding of mathematics and its role in everyday life. Solving real-world problems promotes flexibility and encourages students to adjust and grow their thinking. It inspires them to consider alternatives and apply math in authentic contexts. You will find practical ways to engage students in critical thinking, develop their independence, and make connections with the world.



This remarkable book shows teachers how to stop working harder and start working smarter. It describes a shift from "teach-test-move-on" to "teach-connect-apply" to optimize student learning. This valuable resource provides teachers with an understanding of simple, manageable, and sustainable strategies to change their approach immediately. These strategies build on helping students retain math concepts so they can apply them in real-world situations. The focus is on supporting teachers in framing instruction so that students strengthen their understanding and apply their learning. ***Making Math Stick*** is a game-changer that champions durable learning for all students.

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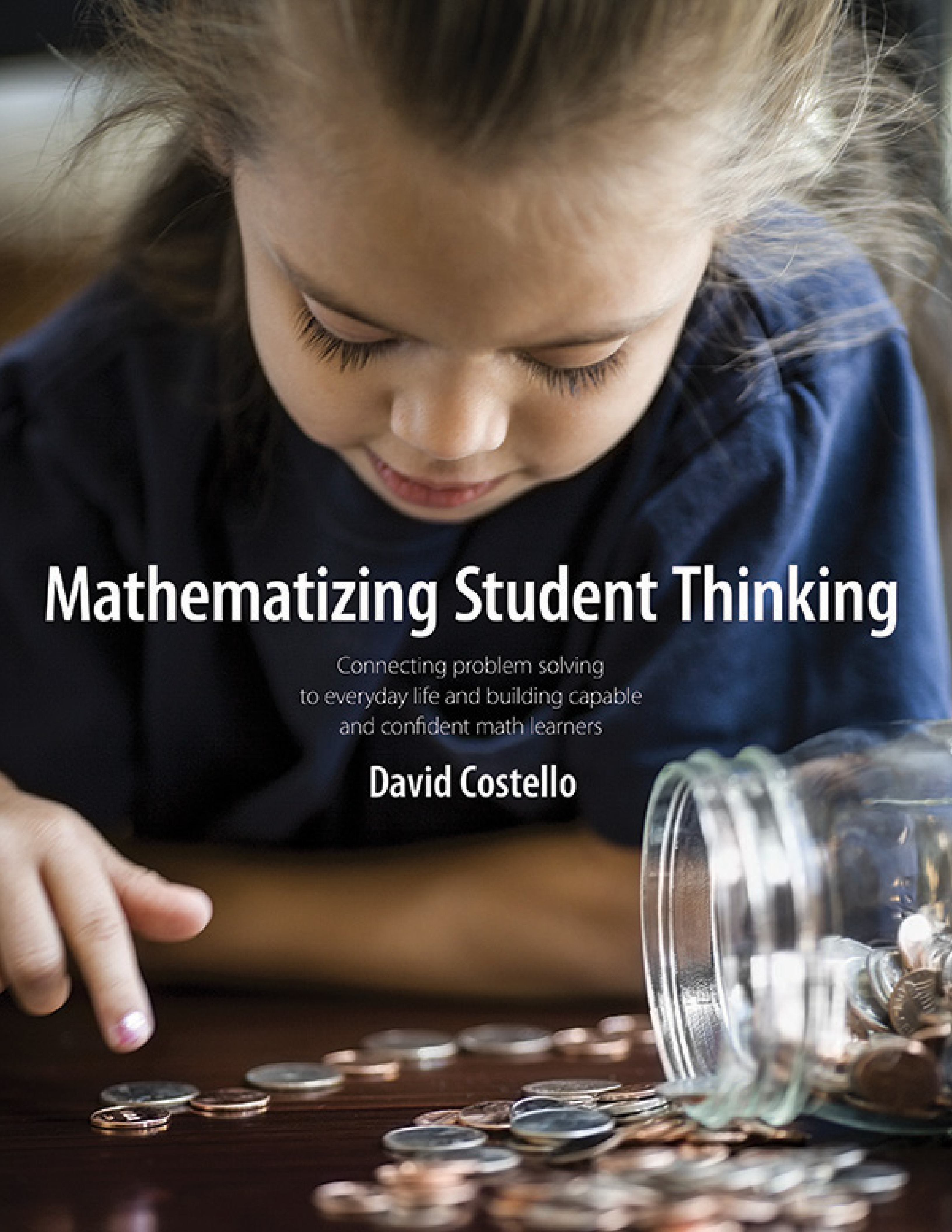


Mathematizing Student Thinking

Connecting problem solving
to everyday life and building capable
and confident math learners

David Costello

Contents

CHAPTER 1

Redefining Problems

When we think about mathematics we tend to think of problems. As stated in the Introduction, our goal is to make students problem solvers. Let's not glaze over the term **problem**. We should not take for granted what a problem means to teachers and students, or how their perspectives differ. Teachers have varied definitions of what a problem entails and have provided some explanations.

THOUGHTS FROM PRIMARY TEACHERS

- "A problem is something that will make students think outside the box. It is something that they haven't seen before."
- "A problem can be anything and everything. What's important is that students don't know the answer right away."

THOUGHTS FROM ELEMENTARY TEACHERS

- "Problems are about students working through stumbling blocks to reach a solution. It's about them persevering and determining how to arrive at a solution that's not initially evident."
- "Problems are about trial and error. They are about people figuring out what is needed and then finding a plan that will get them to the solution."

THOUGHTS FROM INTERMEDIATE TEACHERS

- "A problem is something that requires students to think critically. They need to take time to understand the problem, craft a plan, carry out the plan, and reflect on their work."
- "Problems come in all shapes and sizes. Too often, problems seem to be focused on a number, but this is too limiting. We need to consider problems in all areas of math."

It is apparent that teachers connect problems with critical thinking, perseverance, and thinking outside the box. Now, let's see how students responded to the same question.

THOUGHTS FROM PRIMARY STUDENTS

- "Problems make us think. They are hard."
- "Problems can be just numbers."

THOUGHTS FROM ELEMENTARY STUDENTS

- "We usually get problems at the end of the math block. We use what we learn to figure it out."
- "Word problems are the problems we get. If you look for key words like altogether, share, and groups of, you will know what you need to do to solve the problem."

THOUGHTS FROM INTERMEDIATE STUDENTS

- "Problems are frustrating. Most times problems are about useless things that we don't care about."
- "I used to think problems in math had to have words, but they don't. Sometimes when I get a question with just numbers and have to find the value of the variable, that's a problem."

Clearly teachers and students have different perspectives about what defines a problem. Teachers view problems from the perspective of thinking and perseverance, while students take the view that problems means word problems. The two perspectives are not synonymous. Simply assigning a task with the word "problem" in its title does not ensure that students will problem solve.

What Is a Problem?

Consider whether the following questions. Are they problems or not?

MATH QUESTIONS: PRIMARY EXAMPLES

David has 23 hockey cards. He loses some while cleaning his room. He now has 16 cards. How many cards did David lose?

? = 8 – 2

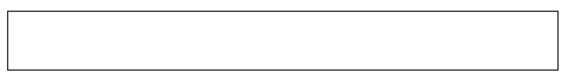
What is the value of the underlined digit? $5\underline{3}8$

What is the missing element?
+ + # + + # + ○ # + +

MATH QUESTIONS: ELEMENTARY EXAMPLES

Find the area.

46 m



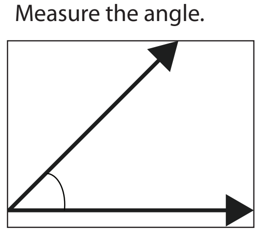
4 m

Compare the following numbers using < , > or =
$\frac{3}{4} \bigcirc \frac{7}{8}$

Find the missing value

x	y
4	9
6	13
3	7
9	?

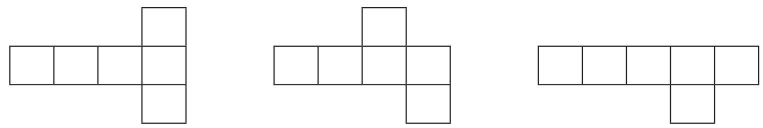
Measure the angle.



MATH QUESTIONS: INTERMEDIATE EXAMPLES

Solve
38 = 6x – 4

Which of the following nets, if any, would produce a cube?



Compare and contrast mean and mode.

Alli places two blue and two yellow marbles in a bag. Find the probability of drawing two blue marbles if the first one is not returned before drawing the second.

Which of the examples did you consider a problem? Why? When you were considering whether the questions were problems or not, did the assigned grade level make a difference?

It is important to recognize that what some people may consider a problem may not be seen as one by someone else. Does previous experience play a role? Is this a novel situation for students, or have they seen it multiple times before?

We must be aware when thinking about what constitutes a problem. There is a distinction to be made between practice and problems.

We must be aware when thinking about what constitutes a problem. There is a distinction to be made between practice and problems. In terms of practice, students approach the question decisively and can easily distinguish what is being asked and find a solution. There is more unknown associated with a problem. A problem is present when the student has a goal but does not know how to achieve that goal.

Let's consider what a problem *actually* is. When examining problems, there are three common characteristics: initial state; goal state; and obstacles (Greenwald, 2000).

Initial State

Initial state is the state that students are presented with at the onset of the problem. For example, in a word problem this would be the scenario that is presented to the student. In an equation, this would be the *what* the student would have to solve to determine the value of the unknown. In a table, it is the information that the student is presented with. In a pattern, it is the sequence (whether repeating, increasing, or decreasing) that students are presented. The initial state is what the student is presented with as the starting point of the problem-solving process.

Goal State

The **goal state** is what is achieved and desired by the student. It is the preferred outcome. For example, in a word problem it is the question that students must be able to answer. In a table of values, it is the pattern that the student must recognize in the values and/or use to identify the unknown value. In a pattern, it may be to recognize the pattern (whether it is repeating, increasing, or decreasing), identify what comes next, and/or correct any inaccuracy in the pattern.

Obstacles

Obstacles, also referred to as stumbling blocks, are what happens between the initial state and the goal state. An obstacle would engage students in **productive struggle**. Initially, students would not be sure how to move from the initial state to the goal state. This may include the student having difficulty understanding the problem, identifying a plan of action to overcome the obstacle, and/or enacting this plan of action. Problem solving is the thinking that happens to work through the obstacles.

Let's take a closer look at the problems we saw earlier putting the three characteristics into practice and examining how they fit into the problems. It is important to note that a student's previous learning is taken into consideration for these problems so they may not be novel. It could be an opportunity to practice instead of problem solving.

PUTTING IT INTO PRACTICE: PRIMARY EXAMPLES

David has 23 hockey cards. He loses some while cleaning his room. He now has 16 cards. How many cards did David lose?

Initial State: David has 23 hockey cards. He loses some while cleaning his room. He now has 16 cards.
Goal State: How many cards did David lose?
Obstacle: Determine the strategy (or strategies) needed and apply it to find out how many cards David lost.

~~~~~~

? = 8 – 2

Initial State: = 8 – 2
Goal State: ?
Obstacle: Determine the strategy needed and apply it to find out the value represented by the ? .

~~~~~~

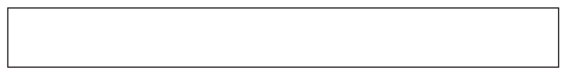
What is the missing element?
+ + # + + # + ○ # + +

Initial State: + + # + + # + ○ # + +
Goal State: What is the missing element?
Obstacle: Determine the pattern and identify the missing symbol.

PUTTING IT INTO PRACTICE: ELEMENTARY EXAMPLES

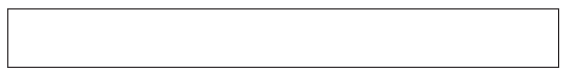
Find the area.

46 m



4 m

Initial State: 46 m



4 m

Goal State: Find the area.
Obstacle: Determine the strategy needed and apply it to find the area.

~~~~~~

Compare the following numbers using < , > or =
$\frac{3}{4} \bigcirc \frac{7}{8}$

Initial State: $\frac{3}{4} \bigcirc \frac{7}{8}$
Goal State: Compare the following numbers using < , > or =
Obstacle: Determine the strategy needed to compare the two fractions; then decide which symbol to record in the circle.

~~~~~~

Find the missing value.

x	y
4	9
6	13
3	7
9	?

Initial State:

x	y
4	9
6	13
3	7
9	?

Goal State: Find the missing value.
Obstacle: Determine the pattern rule (expression) for x to y and then apply this to find the number represented by the ? .

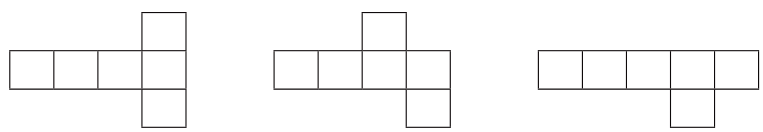
PUTTING IT INTO PRACTICE: INTERMEDIATE EXAMPLES

Solve.
38 = 6x – 4

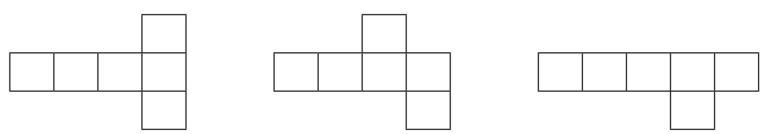
Initial State: 38 = 6x – 4
Goal State: Solve.
Obstacle: Determine the strategy needed and apply it to find out the value represented by x.

\~~~~~~

Which of the following nets, if any, would produce a cube?



Initial State:



Goal State: Which of the following nets, if any, would produce a cube?
Obstacle: Determine if the given nets make a cube; then share which one(s) do.

\~~~~~~

Alli places two blue and two yellow marbles in a bag. Find the probability of drawing two blue marbles if the first one is not returned before drawing the second.

Initial State: Alli places two blue and two yellow marbles in a bag… if the first one is not returned before drawing the second.
Goal State: Find the probability of drawing two blue marbles.
Obstacle: Determine the strategy (or strategies) needed to figure out the probability.

Strengthening the Problem

In the problems we have seen, we notice that there was little connection to the lives of students and few opportunities for engagement and interest. This was intentional to highlight how we can strengthen the problems that we are presenting to students. Consider the following characteristics of **rich tasks**: accessibility; real-life experience; interesting and engaging opportunities for creativity and individuality; variety of approaches and representations; opportunities for collaboration and discussion; and extending learning (Butler Wolf, 2015; National Council of Teachers of Mathematics, 2014; Van de Walle, Karp, Bay-Williams, & McGarvey, 2017).

Accessibility

Accessibility for learners is a significant aspect of rich tasks. Students can come with a wide range of learning abilities and needs. Tasks must be created with multiple possible points of entry and exit so that all types of students can approach the problem from various perspectives, such as mathematical understanding, experience, and foundational knowledge. By having possible **points of entry**, all students can engage in the task and work toward a solution. For possible **points of exit**, tasks that provide students with opportunities to arrive at various responses that could be considered correct. It increases the probability of successfully completing the task.

Real-life Experiences

When the problem can be compared to real-life experiences, students view the task as more meaningful and worthwhile.

When the problem can be compared to real-life experiences, students view the task as more meaningful and worthwhile. When students can recognize the situation and problem, they are more likely to have greater engagement and motivation. This decreases the number of stumbling blocks and lessens the chances of giving up.

Interesting and Engaging

By being able to relate to the problem, student interest and engagement with the problem are stronger. When students are motivated to solve the problem, they feel a need to continue their exploration and discovery. Curiosity about the situation and solution increases and students are eager reach the solution.

Creativity and Individuality

Through an increase in exploration and discovery, the opportunity for individuality is strengthened. Students are encouraged to be creative in their approach to solving the problem. This creativity could be in how they approach the problem, the strategy or strategies they choose as their plan of action, and the action they take. Individuality and creativity used to apply toward mathematical reasoning is as important as the solution itself. The thinking that students apply in the process cannot be understated. It exemplifies their understanding of the problem and the mathematics that they are applying to reach a solution.

A Variety of Approaches and Representations

Problems that lend themselves to a variety of approaches and representations provide an opportunity for students to approach mathematics from a different perspective. By having this openness, there is an increase in the likelihood of more students having success. Through a variety of approaches and representations, students will be confident that there is more than one correct way to approach the problem. This, in turn, encourages students to persevere in determining a plan of action to apply to problem solving.

Collaboration and Discussion

With so many approaches and representations available the opportunity for collaboration and discussion increases among students. By collaborating and discussing mathematics with others, they are strengthening both their understanding of the problem and of other ways to approach a solution. Collaboration and discussion supports metacognition when students must clarify their thinking about the problem and their approach to solving it. They can also reflect on how their thinking is similar to their classmates. Therefore, student thinking is strengthened as is the communicative ability of students.

Extending Learning

The rich task is a problem that provides students with the chance to connect meaning to a familiar lived experience. Relevant problems enable students to engage and work through stumbling blocks and apply creative and individual approaches to solve problems.

A final characteristic of rich tasks that cannot be understated is the opportunity for extended learning. When problems have the openness to allow students the opportunity to extend their learning, mathematical understanding is strengthened. One such example of this extension of learning is being able to make connections amongst mathematical concepts. Through making connections amongst concepts, students build a deeper mathematical understanding as they apply a network of concepts to make sense of the problem and make sense of how to solve the problem. In addition to making connections amongst concepts, students can extend their learning by approaching the problem from a more complex perspective and/or strategy. Through applying a more complex perspective and/or strategy, students can enter the problem from a differing point of entry and can exit the problem from a more complex point of exit. Thereby, students are provided an opportunity to challenge themselves as learners.

The rich task is a problem that provides students with the chance to connect meaning to a familiar lived experience. Relevant problems enable students to engage and work through stumbling blocks and apply creative and individual approaches to solve problems. Rich tasks have multiple points of entry and exit, so there are more ways to apply problem-solving skills. Students can feel more confident talking through their mathematical understanding with their classmates. Now students can move from doing mathematics to thinking mathematically.

Finding Problems

Now that we have considered a problem from the perspective of teachers and students, and have examined what constitutes a problem and a rich task, let's consider where teachers tend to *get* the problems they assign students in the classroom.

THOUGHTS FROM PRIMARY TEACHERS

- "I find it hard to know just what problem to give my students. I usually look in my guides and on the internet to find problems that I can give my class that they will be able to work on and not have to ask me too many questions."
- "When looking, I think if the problem would be meaningful for students and if they could connect to it. Sometimes, I will use the names of my students in the problem to help them make connections."
- "I usually start with one-step problems and then will move to two-step problems. Sometimes, I find that giving problems more complex than this causes my students to be overwhelmed and just give up."

THOUGHTS FROM ELEMENTARY TEACHERS

- "I was told a long time ago that there is no need to recreate the wheel. So, when I need problems, I will look in curriculum, support documents, and program materials. Sometimes it is easy to find the problems to assign, while other times I have to make small revisions to make them accessible for my students. If I can't find problems using those resources, I will search for word problems on the internet."
- "When looking for problems, I will look for word problems that aren't too wordy. I don't want students to have difficulty reading the problems. I am focusing on math not reading."
- "I want word problems that have a structure (start, change, and result unknown). It's important that students experience the different structures of problems."

THOUGHTS FROM IMTERMEDIATE TEACHERS

- "The student textbook is great for problems. I assign the problems in the text because these are the problems they will find on tests and exams. I want them prepared and set up for success."
- "I want problems that have multi-steps (two–to–three) so that students have to think about the process they need to apply to solve the problems. They can't just do the first step and then be finished. They need to know how to use what they figured out to help them in the next part of the problem."
- "It is crucial that students see problems that have information displayed in different ways. For example, students should be able to read information within the word problem, take it from a table, take it from a graph, and take it from a picture. By having these different formats, students will see how information can be presented differently and can strengthen their ability to locate important information."

Regardless of the grade level range above, teachers generally look for problems in the resources they are provided, such as in the curriculum, program materials, and/or course text. When asked for the type of problem teachers select, they were focused on problems that were meaningful, had various structures, and were multi-step. These teachers did have a similar approach. They generally selected from similar sources and chose problems that have varying structures. When considering the variances in problems, all that was referenced was structure and number of steps.

What I want to do next is to situate these responses in the types of problems I have assigned, observed, and/or was told about by educators in primary, elementary, and intermediate grade levels. Using this technique, we will understand more about the types of problems students are assigned and the type of problem-solving experiences students may have in the classroom.

Problems Used in the Classroom

There are many problem types available to assign or share in the classroom. It is important that students experience a variety of problems during their education.

There are many problem types available to assign or share in the classroom. It is important that students experience a variety of problems during their education. I have created a visual that represents the typical problems assigned in the classrooms that I have both observed and worked in over the years.

Procedural — Words-As-Labels — Open-ended — Rich Tasks

I will treat these problem types in isolation so that you have a clear understanding of each type and what they may look like at primary, elementary, and intermediate levels.

Procedural Problems

Once students understand the concepts, they need the opportunity to apply those concepts. This approach to problems is referred to as procedural. **Procedural problems** are ones where the strategy is already identified for students, and they must apply the steps accurately in order to find the solution.

Procedural problems will seem familiar since these were the type of problems we encountered as students during Kindergarten to Grade 8.

PROCEDURAL PROBLEMS: PRIMARY EXAMPLES

Solve:
68 + 15

Solve:
8 = ? – 3

PROCEDURAL PROBLEMS: ELEMENTARY EXAMPLES

Solve:
9 + 6 × 3 – 4

Solve:
178.23 ÷ 6

PROCEDURAL PROBLEMS: INTERMEDIATE EXAMPLES

Solve:
$\frac{2}{3} + \frac{3}{7}$

Solve:
–3 – (-3)

As you can see in the procedural problem examples above, there is still a problematic aspect to the questions. While students are aware of what concept is being applied, they must be able to remember and apply the process accurately.

Consider the primary examples—students must recognize that 68 + 15 is an expression that must be solved. And, in the second primary example, students must recognize that they have to preserve equality and determine that the ? in the equation has to represent 11 for both sides of = to be balanced.

The elementary examples, students recognize that an order of operations must be followed to determine the solution. In the second example, the student must divide 178.23 by 6, but they have a choice on how they approach this division problem.

The intermediate problems, students must solve an addition problem involving fractions and then a subtraction problem involving integers.

There is a place for procedural problems, they just cannot be the only type of problem that students are provided.

While each of the primary, elementary, and intermediate examples shown may not seem to be problematic, they are. Although the concept has been identified for students—conversation of equality, the type of operation, order of operations—students must be able to apply these to the problem to arrive at a solution. There is a place for procedural problems, they just cannot be the only type of problem that students are provided.

The three characteristics of problems are present within these examples: initial state, goal state, and obstacles. To clarify, students are pointed to which strategy to apply to overcome the obstacle. In terms of procedural problems, the obstacle is accurately applying the strategy to reach the goal state.

Words-As-Labels Problems

There are similarities and differences between procedural problems and **words-as-labels**. Both problem types have one solution. Whichever problem is used, there is only one solution that the students are working toward. In this case, students are either right or wrong in their final response.

The main distinction between procedural problems and words-as-labels problems is that there are now words in the problem. Words-as-labels problems provide a context for students. Instead of students being explicitly told which strategy to apply—as they are in procedural problems—students must read through the problem and decide which strategy applies to the situation. Words-as-labels problems are commonly referred to as story problems or word problems.

In these problems, words are simply adding labels to the items to be manipulated. Within procedural and words-as-labels problems, there are limited choices for students in applying strategy. While procedural problems indicate the strategy to be applied, students have some variance in how to approach it. For example, in addition contexts, students can apply facts, count from one or count on to reach the solution. Within words-as-labels problems, students have the same choice once they determine which strategy to apply.

WORDS-AS-LABELS PROBLEMS: PRIMARY EXAMPLES

Elijah had a collection of eight marbles. After looking through some of his brother's old toys, Elijah found seven marbles. How many marbles does Elijah now have in his collection?

~~~~~~

Winnie had three fish in the fish tank. Winnie's family bought her more fish for her birthday. Now, there are nine fish in Winnie's fish tank. How many fish did Winnie's family buy her?

~~~~~~

Juan had some stickers to give his friends. He gave four stickers to his friends and he has eight left. How many stickers did Juan have before he gave any to his friends?

~~~~~~

Elizabeth decided to purchase some items from the store. She purchased a book, poster, and crayons. How much did Elizabeth spend?

Book –$12
Crayons –$8
Poster –$9

~~~~~~

Consider the above problem with extraneous information:
Elizabeth decided to purchase some items from the store. She purchased a book, poster, and crayons. How much did Elizabeth spend?

Book –$12
Markers –$6
Crayons –$8
Poster –$9

~~~~~~

Consider the above problem with an additional step:
Elizabeth had $45 to spend at the store. If Elizabeth purchased a book, poster, and crayons, how much money does she have left?

Book –$12
Markers –$6
Crayons –$8
Poster –$9
~~~~~~

WORDS-AS-LABELS PROBLEMS: ELEMENTARY EXAMPLES

Dane purchased eighteen yearbooks for a cost of $96. If each yearbook was the same price, how much did each cost?

~~~~~~

Talia had seven boxes of books. If each box had 25 books, how many books did Talia have?

~~~~~~

Josef's class had twenty-eight students. Each student raised $50. How much money did Josef's class raise for the fundraiser?

~~~~~~

Consider the above problem with extraneous information:

Josef's class set a goal to raise enough money to purchase 20 soccer balls for the school. Josef's class had twenty-eight students. Each student raised $50. How much money did Josef's class raise for the fundraiser?

~~~~~~

Consider the above problem with an additional step:

Josef's class set a goal to raise enough money to purchase 20 soccer balls for the school. Josef's class had twenty-eight students. Each student raised $50. How much money did Josef's class raise for the fundraiser? If the price of each soccer ball is $40, how much money does Josef's class have left?

WORDS-AS-LABELS PROBLEMS: INTERMEDIATE EXAMPLES

Two friends decided to share a pizza. Ethan ate $\frac{2}{5}$ of the pizza and Elias ate $\frac{2}{4}$ of the pizza. How much of the pizza wasn't eaten?

~~~~~~

At the start of the game, the temperature outside the school was 8°C. At the end of the game, the temperature was −4°C. How much did the temperature decrease during the game?

~~~~~~

Alyah's community decided to have a fundraiser and set a goal of $2500. If the community raised $\frac{3}{4}$ of this goal, how much money have they raised so far?

~~~~~~
~~~~~~

Consider the above problem with extraneous information:

Alyah's community decided to have a fundraiser and set a goal of $2500. If the community raised $\frac{3}{4}$ of the fundraising goal in a three-month period, how much money have they raised so far?

~~~~~~

Consider the above problem with an additional step:

Alyah's community decided to have a fundraiser and set a goal of $2500. If the community raised $\frac{3}{4}$ of the fundraising goal in a three-month period, how much money have they raised so far? How much money does the community still need to raise to meet its goal?

Words-as-labels provide context, but the problem itself can be adjusted to allow for alternate ways of including information in it.

As observed in the previous primary, elementary, and intermediate words-as-labels examples, there are various ways that the problems can be presented. Words-as-labels provide context, but the problem itself can be adjusted to allow for alternate ways of including information. In the primary example, the price of items is listed. This information could be presented in a list or it could have been written out within the sentence of the problem just like the elementary examples. Another way to introduce these options could be to place them in a diagram of each item with the corresponding price attached.

Another way to write words-as-labels problems is to include extraneous information. This would then allow the students to read through the problem and identify what the problem is about, then distinguish between the relevant and irrelevant information. Extraneous information adds another layer of complexity to the situation.

In addition to adding extraneous information, the problem can be adjusted by including an additional step students must address. By incorporating added steps to a problem, students are required to address multiple steps to reach the solution.

All the words-as-labels problem examples include the three characteristics of a problem: initial state, goal state, and obstacles. Within words-as-labels problems, students must decide which strategy to apply and then take this strategy to overcome the obstacle. This, as compared to procedural problems, is an additional layer of difficulty.

Open-Ended Problems

Open-ended problems, also referred to as open questions, approach instruction and learning from a much wider lens than our previous two problem types. Within open-ended problems, students are provided opportunities for a greater degree of choice in the strategy to use and applying it. For example, as can be seen in the examples of open-ended problems, students are presented with a scenario that allows for flexibility in approaching the problem and how they can determine which strategy to apply and how. Simply put, open-ended problems are tasks put together in such a way that opens up possible answers and many ways to get to the answers (Small, 2013).
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The openness of open-ended problems is meant to provide students with multiple points of entry and multiple points of exit. It is about students having choice in how they approach the problem and how they work through it. Such openness is meant to be a source of differentiation so that students at various developmental levels can approach the problem within their zone of proximal development (Small, 2012).

As indicated in the examples provided next, open-ended problems can be presented to multiple student levels and each student can apply their mathematical understanding to the problem so that they can persevere and reach one of the many solutions to the problem.

OPEN-ENDED PROBLEMS: PRIMARY EXAMPLES

What number can you represent using eight base-ten blocks?

~~~~~~

I have two numbers that have a sum of approximately 80. What could the two numbers be?

~~~~~~

Sara is five-years older than her brother. How old could Sara and her brother be?

~~~~~~

A pattern has more green cubes than red cubes. What could the pattern look like?

OPEN-ENDED PROBLEMS: ELEMENTARY EXAMPLES

Which of the following fractions do not belong: $\frac{4}{9}$ $\frac{2}{7}$ $\frac{1}{9}$ $\frac{4}{4}$

~~~~~~

Consider four numbers that are greater than 10 000 but less than 100 000. The sum of the digits in each number is 21. Arrange the four numbers in ascending order.

~~~~~~

The sum of three numbers totalled more than 1000. One of these numbers was more than half of the sum. What could the other two numbers be?

~~~~~~

The perimeter of a rectangle is between 40 cm and 60 cm. The area of this same rectangle is between 100 cm^2 and 120 cm^2. What could the length and width of the rectangle be?

OPEN-ENDED PROBLEMS: INTERMEDIATE EXAMPLES

The product of two fractions is between half and three-quarters. What could the two fractions be?

~~~~~~

When solving an equation, I determine that the value of **x** is 4. What could the equation be?

~~~~~~

Complete the following statement so that it is true:
20% of _______ is ________

~~~~~~

The sum of two decimal numbers is between 10 and 20 while the product of the same two decimal numbers is between 60 and 70. What could the two decimal numbers be?

The various examples of open-ended problems are meant to highlight how students with various levels of mathematical understanding can approach the problem and work through it to achieve a successful response. The openness allows for students to apply a variety of strategies to the problem. The flexibility that is offered with an open-ended problem is the opportunity for students to add meaning and arrive at a response that addresses the question being asked.

As students work through an open-ended problem, they can apply creativity and flexibility to their thinking. They know that there is not only one prescribed way to approach the problem and it is because of this openness that there is more than one available solution. Flexibility promotes creativity, enabling students to feel less restricted in the strategy they choose to solve the problem. In addition, students have a greater likelihood to persevere. When encountering a stumbling block, students are more likely to work through it because they recognize that there are multiple avenues that they can take to get to a solution. Flexibility means that options are available which encourages students to work through the problem.

Sharing strategies and solutions not only supports student reflection and clarity in discovering their own mathematical understanding, but also in how to revise how they think when listening to others' approaches.

A positive aspect of assigning open-ended problems is what it can do for classroom discourse. With multiple points of entry and exit, there is an opportunity for students to share their individual approaches and solutions with one another. By sharing, students will realize that their approach is one of many that can be applied. There is more than one possible solution to the problem they are trying to solve. Sharing strategies and solutions not only supports student reflection and clarity in discovering their own mathematical understanding, but also in how to revise how they think when listening to others' approaches.

Like the other problem types we have discussed, open-ended problems are also comprised of the three characteristics of a problem: initial state, goal state, and obstacles. Students are presented with an initial state and are asked to arrive at the goal state. The emphasis in open-ended problems is that students have more choices in how to overcome obstacles and arrive at the goal state. The goal
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state is quite open since multiple goal states exist. It is not a singular goal state, but instead a goal state that consists of many possibilities.

Rich Tasks

Rich tasks are types of problems that provide students with an opportunity to make meaning within a context that is familiar to their lived experiences. When considering a familiar context, students can recall experiences or situations they have had in and out of school.

Rich tasks are types of problems that provide students with an opportunity to make meaning within a context that is familiar to their lived experiences. When considering a familiar context, students can recall experiences or situations they have had in and out of school. Rich tasks are meant to be opportunities for engagement and interest. Students are afforded the opportunity to select from a variety of approaches and representations as they work toward the solution, where one or more can exist (Butler Wolf, 2015; National Council of Teachers of Mathematics, 2014; Van de Walle, Karp, Bay-Williams, & McGarvey, 2017).

RICH TASK PROBLEMS: PRIMARY EXAMPLES

A Grade 3 classroom of eighteen students had three bookcases. There were three shelves on each bookcase. Each shelf had more books than the number of students in the classroom. How many books were there on the three bookcases?

~~~~~~

Two students were using cube-a-links to make a pattern. One student was making a repeating pattern while the other student was making an increasing pattern. When working on their individual patterns, the students realized that they used the same-coloured cube-a-link for the tenth item in their patterns. Is there a way that they will use the same colour again for the twelfth item in their patterns?

RICH TASK PROBLEMS: ELEMENTARY EXAMPLES

The school soccer team decided to purchase team hoodies. These hoodies were $22 each or 4 for $80. If there were 22 players on the team, how much money would the hoodies cost?

~~~~~~

The local rink had two major programs using the ice surface: figure skating and hockey. Both figure skating and hockey programs agreed to share the ice surface on the first day of each winter month but wanted a schedule that allowed each program to have the ice surface to themselves on other dates. Figure skating was granted ice surface every third day and hockey was granted the ice surface every fourth day. How many dates did each program have to themselves and how many did they have to share?

RICH TASK PROBLEMS: INTERMEDIATE EXAMPLES

At the onset of the school fundraiser, the balance of the account was greater than $100. There were times, over the next eight transactions, during which the balance went up and down. What could each transaction have been and what is the current balance of the account?

~~~~~~

Ali's math course consisted of eight projects that contributed equally to the final grade. The course final grade was comprised only of these eight projects. On the initial project, Ali had their highest grade. What is the average grade for this student?

Within the problems students have opportunities to engage in mathematical discourse with peers.

Interestingly, when examining the rich tasks provided in the primary, elementary, and intermediate examples, you will notice that some may not be as connected to the lives of students out of school. But, all share opportunities for engagement and interest as well as a variety of approaches and representations to apply. Within the problems students have opportunities to engage in mathematical discourse with peers. It is through this engagement that learning can be extended regardless of where students are at in their understanding of mathematics.

As with open-ended problems, students have multiple points of entry and multiple points of exit when working through rich tasks. The presence of multiple points of entry and exit promotes student perseverance as they encounter stumbling blocks. When engaging with rich tasks, students move from a passive stance of *doing* mathematics to an active stance of *thinking* mathematically.

Like the other problem types we have seen throughout the chapter, rich tasks are comprised of the three characteristics found in problems: initial state, goal state, and obstacles. Much like open-ended problems, rich tasks offer flexibility in their approach and number of possibilities for a solution. Where there may be differences between the two problem types is that rich tasks promote a sense of relevancy for students that may not always be present in open-ended problems.

Progression of the Four Problem Types

The four common problem types have been demonstrated throughout the chapter. These are problems observed, applied, and/or discussed with teachers. In previous examples, the four problem types examined various concepts in primary, elementary, and intermediate grade levels. Now, one concept for each grade level band is highlighted to indicate what each problem type could be for a particular concept. This will show how each problem type can strengthen student learning while, at the same time, distinguish between the different thinking points amongst the four problem types.
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PERIMETER PROBLEMS: PRIMARY EXAMPLES

A. Procedural

Determine the perimeter of a rectangle with a width of 3 cm and a length of 8 cm.

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B. Words-As-Labels

A farmer wanted to purchase fencing material for his rectangular field. The field was 70 m long and 40 m wide.

~~~~~~

C. Open-ended

The perimeter of a sign was between 80 cm and 100 cm. What could the length of the sides have been?

~~~~~~

D. Rich Tasks

The Grade 3 class was responsible for making the seasonal display in the school lobby. As part of this display, students had to decorate a bulletin board. The bulletin board had two sides that were much longer than the other two sides. Students wanted to place a decorative border around the bulletin board as it was the centerpiece of their display. What length of border will they need to go around the bulletin board?

DIVISION WITH DECIMALS: ELEMENTARY EXAMPLES

A. Procedural

Solve: 185.49 ÷ 8

~~~~~~

B. Words-As-Labels

David raised $185.49 when fundraising. He wanted to share the money equally between 8 charities. How much money will David donate to each charity?

~~~~~~
~~~~~~

C. Open-ended

When dividing a decimal number by a single-digit whole number, the answer is between ten and twenty. What could the decimal number and single-digit whole number be?

~~~~~~

D. Rich Tasks

The local elementary school was fundraising to help four local charities. Although the students were hoping for a friendly whole number, the total money raised was a decimal number. If the students were hoping to share this money equally, how much money would each charity receive?

MULTIPLY FRACTIONS: INTERMEDIATE EXAMPLES

A. Procedural

Solve:
$\frac{1}{4} \times \frac{3}{5}$

~~~~~~

B. Words-As-Labels

The deck is $\frac{1}{4}$ the size of the garden. The garden is $\frac{3}{5}$ the size of the yard. What size is the deck in relation to the yard?

~~~~~~

C. Open-ended

When multiplying two factions, the product is less than 1 but greater than $\frac{1}{3}$. What could the two fractions be?

~~~~~~

D. Rich Tasks

The intermediate school's volleyball team had a busy schedule of games. During the first half of the season, the team won $\frac{1}{2}$ of the games. In the second half of the season, the team won $\frac{2}{3}$ of their games. How many games did the team win during the season?

While each of the four problem types have similarities, there are significant differences.

While each of the four problem types have similarities, there are significant differences. It is important, however, to note that students benefit from having experience in each type. Procedural problems allow students to focus on the process and to devote their energy to applying the process. It is a great exercise once students understand the concept. Words-as-labels problems provide students opportunities to engage with the concept and apply context to them. They can read the problem, determine the strategy to take, and then apply it to the concept. It is an additional layer of thinking for students. For open-ended problems, students have a greater degree of choice and are able to enter the problem from an accessibility point of view. Similarly, rich tasks provide opportunities for students to have choices but also provide the concept in a situation that relates to their lived experiences.

All four problem types play a part in student learning. It is an instructional decision when to assign such problem types and which problem types students will engage with most often. Guidance through this instructional decision is how students think mathematically as opposed to doing math by rote.

Defined Problems

The following is a visual highlighting the four problem types we have discussed.

Procedural   Words-As-Labels   Open-ended   Rich Tasks

The visual represents the typical problems assigned in the classroom that have been observed and worked on for many years. Each problem type falls under the defined problem category.

Defined Problems

Procedural   Words-As-Labels   Open-ended   Rich Tasks

Each of the four problem types are examples of **defined problems**. As previously stated, defined problems include initial state, goal state, and obstacles. A central component of defined problems is that all the information students need to solve the problem is initially present (English, Fox, & Watters, 2005). While this may seem obvious to how all problems are structured, it is important to take a step back and reflect. Students are given a great deal of information and their ability to interpret has either been minimized or eliminated. Although the cognitive demand required to solve the problem increases as we move from left-to-right on the visual, students are still handed much, if not all, of the information necessary to overcome the obstacle and solve the problem.

Now, let's consider how often this happens in our everyday experiences outside of school. Very rarely are we handed all the information needed to overcome obstacles and solve the problem. Instead, we pause, interpret, and work through a situation to locate the information required to solve a problem. It is more of an active stance to learning.

So, in essence, if we are only providing students with defined problems, are we doing enough to move students from *doing* math to *thinking* mathematically?

So, in essence, if we are only providing students with defined problems, are we doing enough to move students from doing math to thinking mathematically?

Ill-Defined Problems

What I want to advocate is that we need to move further to the right if we want students *thinking* mathematically. The following visual shows the addition of ill-defined problems to the right-side.

Defined Problems				Ill-Defined Problems
Procedural	Words-As-Labels	Open-ended	Rich Tasks	Ill-Defined

Ill-defined problems tend to be less common in the classroom but can be a game-changer for teachers wanting to reach their instructional goal of making mathematics relevant and having students *think mathematically*. One major distinction between defined problems and ill-defined problems is that ill-defined problems are missing one or more of the three characteristics of a problem: initial state, goal state, and obstacles. The purpose for missing one of more of these characteristics is to have problems that resemble ones that can be encountered in the real world.

Within ill-defined problems, the problem solver plays a significantly independent role. Typically, the initial state is vague and limited. Students must recognize that information is not as forthcoming as it usually is in defined problems. In addition, ill-defined problems rarely provide students with a specific goal state. Instead, the goal state is influenced and therefore determined by the problem solver. Within such a role, engagement is heightened as is the autonomy students have as they enter, work through, and solve the problem.

Let's consider the solution regarding ill-defined problems. While there are opportunities to have a range of solutions within defined problems (consider open-ended and rich tasks), there is an increased emphasis in the ability of students to arrive at a novel solution in ill-defined problems. As students have a more active presence in understanding the initial and goal state, there is greater variability in the possible solutions. This will then increase the likelihood for multiple paths toward solutions to exist.

There is the possibility to have multiple solutions in ill-defined problems because these problems tend to be complex and poorly defined. Solutions to ill-defined problems are based on the interpretations students make while working through a vague initial state and determining the goal state (Byun, Kwon, & Lee, 2014). What the student brings while interpreting the problem will shape whether their response makes sense or not. As such, you can see how students are autonomous and independent when engaging with ill-defined problems.

What the student brings while interpreting the problem will shape whether their response makes sense or not. As such, you can see how students are autonomous and independent when engaging with ill-defined problems.

Ill-defined problems are unclear and require students to question what is known, what needs to be known, and how an accurate response to the problem can be achieved (Greenwald, 2000). Such ambiguity in the problem creates several pathways to multiple solutions. This multiplicity is influenced by the student's mathematical understanding and lived experiences. Therefore, instead of being the recipients of the information while working on defined problems, students become active, independent learners as they interpret vague situations and use their understanding and experiences to make meaning.

The following are examples of ill-defined problems for Primary, Elementary, and Intermediate levels.

ILL-DEFINED PROBLEM: PRIMARY EXAMPLE

The Kindergarten classroom recognized that many of their supplies for centres were needing to be replaced. The teacher shared that they needed to buy new supplies but that they had a limited budget.

ILL-DEFINED PROBLEM: ELEMENTARY EXAMPLE

The local elementary school was selected as the site for the cross-county meet. As part of holding this meet, the school had to decide how to accommodate people wanting to watch the meet.

ILL-DEFINED PROBLEM: INTERMEDIATE EXAMPLE

In planning end-of-year activities, the school administration decide that the Grade 8 students would go on a field trip. The administration wanted to know what the cost per student would be for such a trip.

Ill-defined problems provide students with opportunities to engage in thinking mathematically instead of doing math. Ill-defined problems closely align to the problems students are faced with outside of the school setting.

Ill-defined problems provide students with opportunities to engage in *thinking* mathematically instead of *doing* math. Ill-defined problems closely align to the problems students are faced with outside of the school setting. Rarely outside of school do students encounter problems where all information is readily available and handed over to them. Instead, students need to examine the situation, interpret what is known and what is not known, and then take an active stance to working toward a possible solution. By engaging in ill-defined problems, students experience mathematics that resembles their lived experiences outside of school. They take an autonomous stance to their learning and work through obstacles that are similar to what they would need to work through to solve problems outside of school. It is about seeing math around them and applying such math to problems that they encounter. Ill-defined problems move students one step closer to being able to *think* mathematically.

Similarities and Differences between Defined Problems and Ill-Defined Problems

Throughout the chapter you have seen that there is a time and place for all problem types in the classroom. Each type of problem has a specific purpose. What we, as teachers, need to be aware of is the reasons we are choosing the problems we assign. To come to this conclusion we must understand the similarities and differences between defined problems and ill-defined problems.

The tables proceeding indicate the similarities and differences between defined and ill-defined problems. Let's examine the foundational aspects of each problem.

Three Characteristics of a Problem	
Defined	**Ill-Defined**
Clearly defined characteristics: • Initial state • Goal state • Obstacles	Missing one or more characteristics: • Initial state • Goal state • Obstacles

Open-ended	
Defined	**Ill-Defined**
• Not all defined problems are open-ended. • Procedural and words-as-labels are not open-ended. • Other examples of defined problems are open-ended.	All ill-defined problems are open-ended.

Rich Tasks	
Defined	**Ill-Defined**
• Procedural problems, words-as-labels problems, and some open-ended problems are not rich tasks. • Other defined problems, such as certain open-ended problems and rich tasks, fit these characteristics.	All ill-defined problems would be based within the characteristics of a rich task.

Multiple Strategy Options	
Defined	**Ill-Defined**
• Procedural problems only have one strategy option. • Words-as-labels may have multiple strategy options, but this is not a certainty. • Open-ended problems and rich tasks have multiple strategy solutions.	All ill-defined problems have multiple strategy options.

Multiple Solutions Available	
Defined	**Ill-Defined**
• Procedural problems and words-as-labels problems do not have multiple solutions available. • Open-ended problems have multiple solutions available. • Rich tasks may or may not have multiple solutions available.	All ill-defined problems have multiple solutions available.

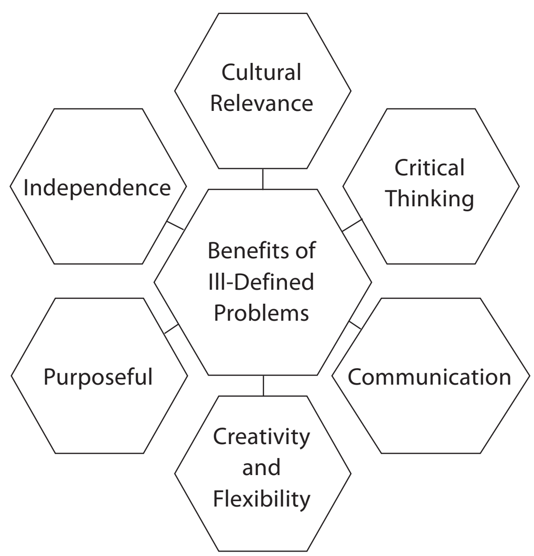
Real World Context	
Defined	**Ill-Defined**
• Procedural problems and words-as-labels problems do not have real world context. • Open-ended problems may or may not have real world contexts. • Rich tasks may or may not have real world contexts.	All ill-defined problems has real world context.

Ill-defined problems are about students thinking mathematically instead of simply doing traditional math problems.

Each table shows the key differences between defined and ill-defined problems. For ill-defined problems, there is more ambiguity within the problem, creating multiple options for crafting a plan to overcome obstacles and for a solution. Such an approach to mathematical thinking allows for learning experiences in the classroom and everyday experiences outside of school to be more relevant. Ill-defined problems are about students *thinking* mathematically instead of simply *doing* traditional math problems. The thinking that students do when working through ill-defined problems is much more mathematized that the *doing* mathematics students do when working through defined problems.

Importance of Ill-Defined Problems

There are several advantages of assigning ill-defined problems in the classroom. We will explore these more in subsequent chapters but first let's understand why they are important. The following is a visual representation outlining the benefits of having students work through ill-defined problems.



The visual identifies the benefits of having students work through ill-defined problems. Ill-defined problems are rooted in real-world experiences and, as such, present students with **culturally relevant** and meaningful problems.

Students must approach their work independently and use critical thinking skills. Students take an active learning stance to such problems in that they must generate aspects of the problem that provide structure. Students encounter a variety of stumbling blocks that require creativity and flexibility to solve the problem. And, with there being multiple points of entry and exit within ill-defined problems, students must be able to clearly communicate their thinking and the many steps they took to find a resolution.

Why Don't We See More Ill-defined Problems?

After hearing the benefits of assigning ill-defined problems in the classroom, you may be wondering why we don't see more ill-defined problems in the classroom. Consider the following comments made by teachers when asked why they don't use more ill-defined problems in the classroom.

THOUGHTS FROM PRIMARY TEACHERS

- "These types of problems are too hard for students. Students have difficulty with word problems. How can I expect them to have success with ill-defined problems?"
- "I can't imagine how you would go about assessing student work on ill-defined problems. There's just too many grey areas. How can I give students descriptive feedback if the problem is so open? It isn't just about if they are correct or not. But I would have to dig a lot deeper, and I am not sure what that would look like."

THOUGHTS FROM ELEMENTARY TEACHERS

- "I use the resource that I am directed to use. And in this resource, there are only defined problems. The problems are meant to support the learning of the unit. Students need this scaffold approach so that they can apply what they are seeing in the lesson. Plus, even if I want to do more ill-defined problems, where would I ever find them? I can't use something that I can't find."
- "My students take standardized tests, and I want them to do well on those tests. There are no ill-defined problems on any standardized test that I have ever given my students. So, why would I give them problems that they won't see again."

THOUGHTS FROM INTERMEDIATE TEACHERS

- "I try to show students how math is all around us. I can do this with word problems. I don't need ill-defined problems because they are just too much for students. I don't want students to be overwhelmed."
- "When students have standardized exams, they don't see ill-defined problems. They see defined problems. I want to provide opportunities for students to grow in their ability to solve problems that they will see later in the year. I want students to be able to experience success on problems that they need to do well on when taking standardized tests."

In considering the question of why we don't see more ill-defined problems in school, there are five underlying reasons:

1. Lack of Experience

Let's consider how many times we have been assigned an ill-defined problem while at school. Perhaps you found that there were none or very few ill-defined problems presented when you were the student. If we have had experiences of ill-defined problems it will likely be from when we were students in senior grade levels or postsecondary. Because of these experiences, we may have subconsciously attributed ill-defined problems to being something students work on in high school and/or college. However, we need to move away from this mindset. The benefits shared in the previous section can be applied to all students in Kindergarten through Grade 8, and we need to appreciate this.

2. Difficult to Create

Let's say that we do want to assign these problems in our Kindergarten through Grade 8 classrooms, now we have to find and/or create ill-defined problems. This can be challenging. There are more defined problems available to teachers than ill-defined problems. The lack of resources may hinder teachers from assigning them in the classroom. In terms of creating ill-defined problems, consider the complexity that may be involved. If we are not used to working with ill-defined problems and if we cannot find many to read through, the thought of creating some from scratch can be overwhelming.

3. Difficult to Assess

Now, let's say that we have an ill-defined problem ready to go, whether it is one we found or created. We have to consider how we are going to assess it. When assessing defined problems, there is less ambiguity because there is less openness. Traditionally, there is one solution and one or a few strategy options available to arrive at the solution. This is not the case for ill-defined problems. There is openness in the interpretation of the problem, in the plans that students can create to lead to one of many possible solutions. What this openness does is move assessment from being about following a prescribed approach to being flexible, fluid, and responsive. As teachers, we must appreciate this openness in student thinking and be able to account for it whether it is through conversation, observation, and/or product.

4. Standardized Tests

Think back to the standardized tests that you have seen in the past or the standardized tests that you have seen recently. Many, if not all the problems are defined. There are a variety of defined problems (procedural, words-as-labels, open-ended, rich tasks), but there are little to no ill-defined problems. When teachers observe that standardized tests do not contain ill-defined problems, they may be less motivated to offer these in class. Teachers want their students to do their best on the standardized tests and as such, focus on problem types that they will encounter on such tests. As a result, we see few ill-defined problems in the classroom.

5. Moving from Product to Process

Ill-defined problems enable students to mathematize their thinking and to become capable and confident math learners. Moving from product to process highlights the intricacies in student thinking and moves learning from being teacher-directed to student-led.

When considering openness, we must consider how moving from defined problems to ill-defined problems moves the focus of mathematics from product to process, and from *doing* mathematics to *thinking mathematically*. Ill-defined problems enable students to mathematize their thinking and to become capable and confident math learners. Moving from product to process highlights the intricacies in student thinking and moves learning from being teacher-directed to student-led.

As can be seen from the teachers' comments there are similarities as to why they do not assign ill-defined problems in the classroom. However, what we are doing is removing one category of problems from students' mathematical experiences. Removing this category means we are eliminating any opportunity to strengthen students' ability to work through ill-defined problems. Students' mathematical thinking is restricted when they are given problems that come with all the needed information to solve the problem. This lessens students' autonomy and independence as mathematicians. When we remove ill-defined problems from the classroom, we increase the isolation of mathematics from the everyday experiences' students have outside of school.

Summary

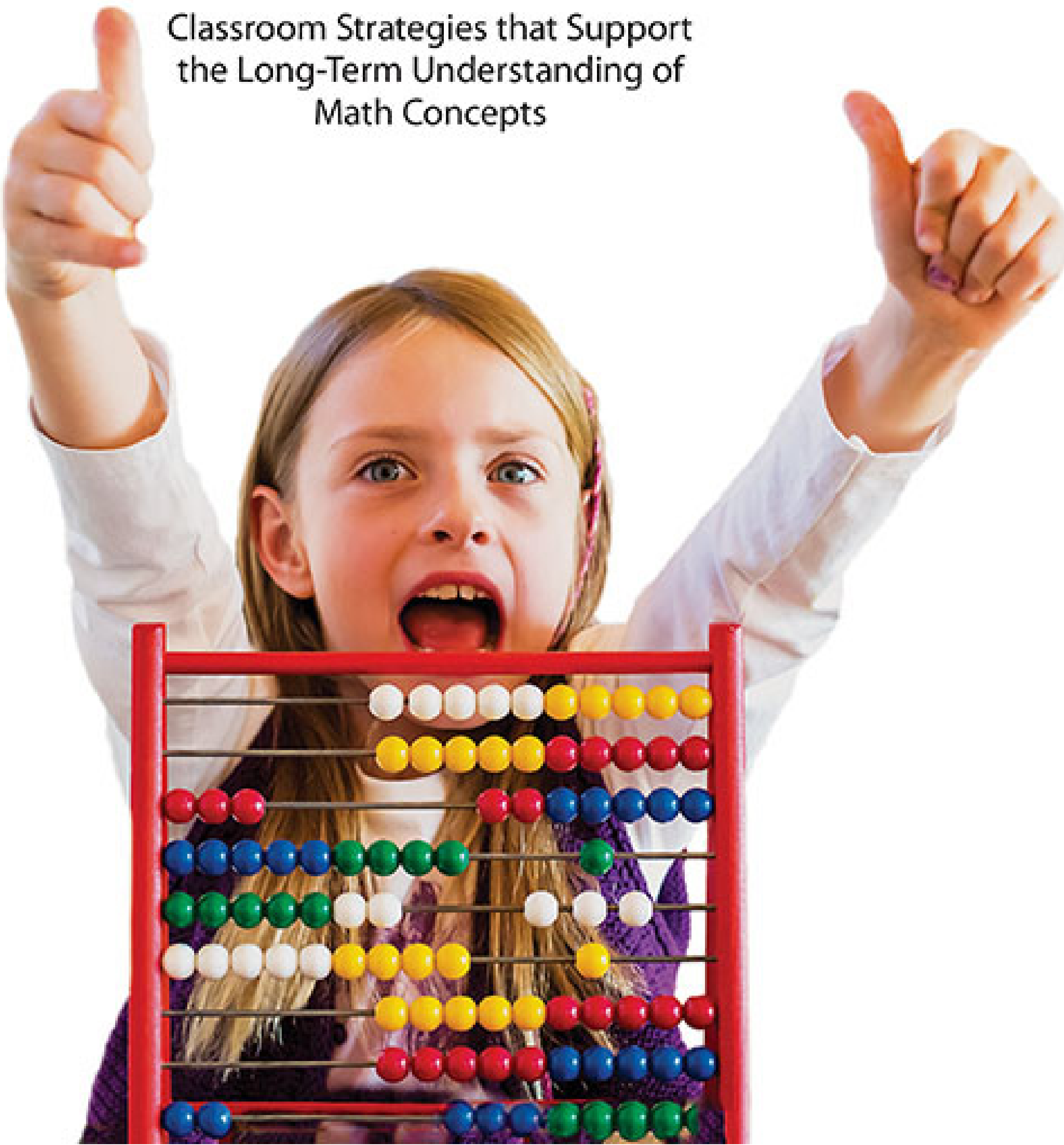
We hear countless references to problems when talking about mathematics. However, it is important to take the time to consider the perspectives of different problems. Problems are characterized by three aspects: initial state, goal state, and obstacles. When examining the problems assigned in many classrooms, it is mostly defined problems that students encounter. While there is a range within defined problems—procedural problems to rich tasks—all the information required to solve the problem is provided. These are not problems we encounter in the real world.

As educators we need to consider and offer students more experiences and working with ill-defined problems will do this. Within such problems, students encounter messy situations that reflect everyday experiences. There are benefits to assigning ill-defined problems, so let's make the switch.

DAVID COSTELLO

MAKING MATH STICK

Classroom Strategies that Support the Long-Term Understanding of Math Concepts



Contents

CHAPTER 1

Learning through Self-Assessment

Help your students use objective tools to gain a realistic perspective of what they don't know yet and to consolidate what they do know.

When we are on a journey, it is important to have a clear idea of our starting point, our destination, and our means of transport. There is no point in rushing out the door unless we know in which direction to travel. And, as we travel, it helps to know where we are, so that we can stay on track. What we need is a global positioning system (GPS).

Too many students appear to be lacking a GPS for their learning journey.

Too many students appear to be lacking a GPS for their learning journey. They may identify, or be told, what the learning goal is for a particular lesson or unit but be unsure where they are in relation to that goal. Students can better plan their route to mastery of a concept if they know what they know about the concept and what they do not know yet. This understanding must be objective and not based on inaccuracies. For students, the GPS they need for learning is effective self-assessment. Through ongoing self-assessment and feedback, students can gain a clear and accurate picture of their status and what their next steps should be.

Self-assessment (also referred to as calibration) is the act of using an objective tool to gain a realistic perspective of what we know and what we do not know (Brown, Roediger III, & McDaniel, 2014). By aligning our self-assessment with *objective* feedback, we avoid illusion of mastery.

To make self-assessment useful, we must guide students in using objective assessment tools. Using an objective tool enables students to avoid illusion of mastery because—like all of us—students are susceptible to inaccurate notions of what they know. When you provide students with objective tools, you will be helping them to get an accurate picture of where they stand.

To be effective, self-assessment tools must also provide timely feedback. Too often, when studying or practicing a strategy, students will move forward through countless problem sets without identifying if or where they went wrong. What this does is create two fallacies. First, without knowing if their responses are correct or incorrect, students will not have an accurate gauge of their understanding. They may think that they are strengthening their understanding of a concept when in fact they are not. What they are doing, instead, is reaffirming an inaccurate application of a strategy. Second, without checking their responses for accuracy, the student is forming a path not toward their learning goal but away from it.

Thus, to strengthen their ability to apply previous learning, students need assessment tools that provide feedback that is both objective and timely. Objective self-assessment tools can include self-quizzing, flashcards, study guides, question sets, self-monitoring, and so on.

Timing is key to successfully using self-assessment tools for learning. Taking a self-assessment directly following an initial lesson is not terribly helpful. The point should not be to measure short-term performance. Self-assessing is most useful when we take stock of our status after some time has passed, to assess what learning has "stuck."

Retrieval practice that is spaced and mixed provides students with the best opportunity to accurately self-assess. When time has passed and concepts are mixed, the learner is forced to think about the prompt, retrieve previous learning, reconsolidate knowledge, and then apply it to answer the question. This form of self-assessment will inform students of their progress in relation to their learning goals. It's about taking stock of their *retention* of understanding.

Self-assessment tools are useful not only for *measuring* learning but for making that learning happen.

Self-assessment tools are useful not only for *measuring* learning but for making that learning happen. While using a self-assessment tool may take more effort than rereading texts or notes, the greater effort exerted by the student will strengthen both the pathways to previous learning as well as the previous learning itself.

By self-assessing, students give themselves immediate feedback that can guide them in next steps. This may involve focusing on certain topics or changing the type of questions to practice. Self-assessment is an accurate guide for students to support them in becoming independent learners.

Some students may be able to create and answer questions that focus on the main ideas of a concept. Others will struggle to create suitable questions, so you may want to supply questions to prompt their thinking. After completing the self-assessment and identifying weaknesses, students can then address the gaps in learning with additional work.

Self-assessment tasks such as self-quizzing are powerful because students must work through desirable difficulties to recall previous learning and apply it to new contexts. To make this work, these tasks must be spaced over a period of time and must mix up the tasks being reviewed so that the same strategy isn't the focus of a single self-assessment experience.

PRIMARY EXAMPLE: A STUDENT EXPLAINS

Using self-assessment to learn addition facts

I would work on the facts at school and at home. When I was asked questions in class, I couldn't remember my facts. I started to use the flashcards for addition and would put my work into two piles—what I knew and what I didn't know yet. If I kept answering a flashcard right, maybe like three or four times, I would put it into the "I know" pile; if I didn't know a flashcard right away, I would put it into the "don't know yet" pile. Then I would practice the "don't know yet" pile more than the "I know" pile. That helped me spend more time on the facts I didn't know.

I still practice both piles, but I practice the "don't know yet" pile more because that is what I need practice with.

Building Retention: This chapter introduces the following examples of Learning through Self-Assessment, which you can help your students use to build their retention capabilities:

- create their own **self-quizzes**
- use **flashcards** for mixed and spaced review
- **self-monitor**

Students Create Their Own Self-Quizzes

Suggested Prompts

To inspire students as they create their own self-quizzes:

- ***How could you decide which concepts to focus on?***
- ***What type of questions could you ask yourself?***
- ***What would you do if you responded to a question incorrectly?***

You may have to work with students to help them be comfortable responding to prompts like those above. For example, consider having students share their responses to these prompts so that classmates can hear other students' thinking. This is especially important as many students may not have encountered this approach to learning, or may not have a lot of experience with it.

What does "studying" look like? Most people would say it involves reading over notes and previous work. Self-quizzing is a different way to study that provides students with information that reading notes does not. Students may find the idea of self-quizzing unappealing because it requires more effort than reading over notes (Brown, Roediger III, & McDaniel, 2014). It is this additional effort, however, that strengthens both consolidation and retention.

During the process of self-quizzing, students must search their long-term memory for previously learned knowledge that relates to the specific questions being asked. What this does is strengthen that previous learning as students add context to the memory in the process of consolidation. It also strengthens the pathways to this learning.

For self-quizzing to be effective, students need an opportunity for immediate feedback. Feedback can take the form of the student checking their own work or having another student check their responses. This feedback will confirm to students if they were able to recall and apply previous learning. Feedback will help them gain an accurate picture of what they know and what they do not know. Students will then have a much better awareness of their progress and what next steps to take than they would have had if they only reread their study notes.

Depending on your students, you can provide an initial set of questions. If possible, though, have students design their own self-quizzes. Doing so will require them to think about which information to focus on. This creates an opportunity for them to sort through topics and identify the more prevalent ones. They can achieve this by either crafting their own questions or selecting questions from a text.

D

DIGITAL SOLUTIONS

Students could create self-quizzes using digital forms.

It will be important for you to support your students in developing their self-quizzing skills. Facilitate discussions on selecting content, question types, and the importance of validating responses. To be effective in self-quizzing, students will need time and support. Students may be tempted to choose questions they can answer easily. These would not provide them with the desirable difficulty that is a significant part in the learning process. Support these students in developing more complex questions that can give them a clearer snapshot of their learning.

The following three examples highlight how students can strengthen their ability to retain previous learning by self-quizzing and then getting feedback.

PRIMARY EXAMPLE: STUDENT WORK

Self-quizzing counting by 2s

A student rolls two number cubes and then counts from that number by 2s. She writes down the number sequence and then uses a hundreds chart to check if her response is correct.

ELEMENTARY EXAMPLE: STUDENT WORK

Self-quizzing two-digit multiplication

A student creates multiple two-digit by two-digit multiplication problems to solve. He solves the problems and then compares his work to the result found using a calculator.

INTERMEDIATE EXAMPLE: STUDENT WORK

Self-quizzing key term definitions

A student records five key words from a math unit previously studied. He writes a definition for each from memory. The student then compares his written definitions with those in his notes to check for accuracy.

Students Use Flashcards for Mixed and Spaced Review

Suggested Prompts

To inspire students as they use flashcards for mixed and spaced review:

- ***How often could you use your flashcards?***
- ***How can you check if your response is correct?***
- ***How will you decide which questions to keep working on?***
- ***How will you know when to move on from a question?***
- ***How will you organize your flashcards?***

Did you ever use flashcards to rote learn your multiplication facts? Many of us did. And yet, flashcards can be used for so much more. The trick is in *how* we use flashcards. By modelling a variety of ways to use flashcards in the classroom, you can emphasize the effectiveness of flashcards as a learning strategy. You can help students see that flashcards can help them consolidate and retain a wide variety of previous learning. Flashcards can focus on vocabulary, math facts, equations, graphs, patterns, and so on.

First, students can use flashcards to space review opportunities over a length of time. It's the kind of tool that is easy to return to time and again. This recurring retrieval of previous memories strengthens the pathways to knowledge.

Second, students can mix up the flashcards so that consecutive cards do not require the same strategy to solve the problem. When flashcards are unorganized, the review is mixed, thereby increasing cognitive demand for students. When seeing a new flashcard, students must pause to analyze the problem, consider which strategy is appropriate, and then apply their chosen strategy.

By setting aside flashcards they have answered correctly, students can focus on the questions and problems that they find more challenging. They should not, however, drop flashcards out of rotation too quickly. Answering a flashcard correctly once or twice does not guarantee that they will be able to recall this information in future. Students should answer a flashcard correctly at least three times, spaced over a period of time, to ensure that the memory isn't just a short-term memory (Karpicke, 2009).

D

DIGITAL SOLUTIONS

You can create a digital flashcard set by making a slideshow, with each slide being a flashcard.

Typically, teachers have been the ones who have created the content for flashcards. This has its benefits, as the teacher can ensure all aspects of the concept are included. Having students create their own flashcards, however, has different benefits. During creation of the cards, students will have to access their previous learning for the finer details of the concept and find creative ways to apply it. Furthermore, these custom flashcards can be a classroom resource that is naturally differentiated to meet the learning needs of students.

Here are a few sample topics that can be addressed using flashcards:

- **Primary**—comparison, addition, subtraction, before/after, subitizing, patterns (increasing, decreasing, and repeating), and shapes
- **Elementary**—addition, subtraction, multiplication, division, place value, measurement (area, perimeter, volume), fractions, integers, and angles. Also relating decimals, fractions, and percentages.
- **Intermediate**—addition, subtraction, multiplication, division of fractions and integers, order of operations, solve for the unknown, and geometry

Flashcards can be used in math games such as Sort, Solve Snap, Solve War, Explain, Memory, Order, and Match. The following three examples illustrate how flashcards can be used effectively.

PRIMARY EXAMPLE: STUDENT WORK

Using flashcards to practice subitizing

A student used a set of flashcards with a ten frame on one side and a number on the flip side. After subitizing the number of dots on a flashcard showing a ten frame, the student checked the number on the back of the card.

ELEMENTARY EXAMPLE: STUDENT WORK

Using flashcards to practice multiplication facts

A student uses a set of flashcards that have a multiplication expression on the front and the product on the back. When the student responded correctly, he placed the flashcard in a "got it" pile. When he responded incorrectly, he placed the flashcard in a "not yet" pile. He continued to practice both piles, but the "not yet" pile would be practiced much more in comparison with the "got it" pile.

INTERMEDIATE EXAMPLE: STUDENT WORK

Using flashcards to work with math terms and tasks

A student had two colors of flash cards. Green cards had a math term on the front and a definition on the back. The student went through the pile of cards one by one. For each math term, she recalled the definition and then looked at the back of the flashcard to check her response for accuracy.

Blue cards each had a task on the front of the card and a solution on the back. The student worked through the task and then compared her response to the solution on the back of the card.

Students Self-Monitor

Suggested Prompts

To inspire students as they self-monitor:

- ***Try talking out loud as you work through the problem: say what part you get and what parts you don't get.***
- ***How do you know when you hit a stumbling block?***
- ***How will you know if you are making meaning?***
- ***How will you check to see if you're right?***

In too many classrooms, students ask their teachers, "Is this right?" or "How do I do this problem?" When the teacher answers such questions, it is the *teacher* who is thinking. By teaching students to self-monitor, you can empower *them* to do the thinking. You give students an ongoing, live feedback of their progress. They no longer must rely on you to tell them if they are right or not. It is the student monitoring their own thinking—a crucial aspect of independent learning.

Students self-monitoring for comprehension means that they are continually checking to see if their work makes sense as they work through a problem or task. It situates the student as the thinker. Self-monitoring students who run into road-blocks work to pinpoint if their confusion stems from the problem as a whole, from only part of the problem, or from an uncertainty about what approach to take. They try to identify the ideas, concepts, or themes that do not make sense.

Once self-monitoring students have identified areas of difficulty, they can then select an appropriate strategy to repair their gap in comprehension. They can turn to a variety of "fix-it" strategies, such as rereading the problem, making sure that all relevant information has been identified in a problem, checking to see if the strategy is being applied appropriately, and determining if the initial response even makes sense.

To help your students get the idea of self-monitoring, model the practice: as you demonstrate solving a problem, talk out loud and describe your thinking process. Purposely make mistakes and then talk yourself through your work to identify the errors. Explain that this is how you would like them to talk to themselves while they do problems. If your students need help getting comfortable with the practice, have them begin by trying to explain their thinking to you or a partner.

The following examples show students talking through their thinking or describing their strategies for self-monitoring as they work through a problem.

PRIMARY EXAMPLE: STUDENT WORK

Self-monitoring skip counting

I'm skip counting by 5s to 100. I'm starting by saying 10, 15, 20 … Now I just said a number that ends with an 8. I know that doesn't make sense because, when you count by 5s starting at 10, the last part of the number should always be 0 or 5.

ELEMENTARY EXAMPLE: STUDENT WORK

Self-monitoring work on geometry problems

I'm working with translations, rotations, and reflections…. I'm not sure if my rotations are done right. Should I ask the teacher? No, I want to do it myself.… So, I'll start by putting a question mark beside anything that I'm not sure of and an exclamation mark beside things that I know for sure…. Okay, now I'll go back to the question marks and think about it a little more…. I can make some small changes…. Good, that's right now.

INTERMEDIATE EXAMPLE: A STUDENT EXPLAINS

Self-monitoring work on fractions

I was solving some equations with fractions, and I wasn't sure if I was on the right track with my work. So I stopped and went back to the beginning of my work and rechecked all of my steps. I saw that everything was correct, so I kept going and solved the problem. I wanted to be sure that I was correct and not making any mistakes along the way.

Learning through Self-Assessment Summary

Each of the strategies highlighted within Chapter 1—creating and taking self-quizzes, using flashcards for mixed and spaced review, and self-monitoring—are different ways of Learning through Self-Assessment. They can all play a significant role in helping students gain an accurate understanding of what they know well and what they do not know well. Further, all three strategies can help consolidate previous learnings. Instead of considering self-assessment as a measuring activity, the three strategies situate self-assessment as metacognitive activity that strengthens learning.

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